Math 2374
Spring 2010
Final
May 10, 2010
Time Limit: 3 hours

Name (Print): ________________
Student ID: ________________
Section Number: ______
Teaching Assistant: ________________
Signature: ________________

This exam contains 12 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	35 pts	
3	35 pts	
4	30 pts	
5	35 pts	
6	30 pts	
7	30 pts	
8	25 pts	
9	30 pts	
10	25 pts	
TOTAL	300 pts	

1. (25 points) Compute an equation for the plane tangent to the graph $z = f(x, y)$ of

$$f(x,y) = \frac{e^x}{x^2+y^2}$$

at $x = 0$, $y = 1$, $z = f(0, 1)$.

Solution: Approach 1: $\frac{\partial f}{\partial x} = \frac{e^x}{x^2+y^2} - \frac{2xe^x}{(x^2+y^2)^2}$ At $(x,y)=(0,1)$, $\frac{\partial f}{\partial x} = 1$

$\frac{\partial f}{\partial y} = \frac{-2ye^x}{(x^2+y^2)^2}$ At $(x,y)=(0,1)$, $\frac{\partial f}{\partial y} = -2$. Also $f(0,1) = 1$.

The tangent plane is $z = (x-0) - 2(y-1) + 1$ or $x - 2y - z + 3 = 0$.

Approach 2: Parametrize the graph: $\Phi(u,v) = \left(u, v, \frac{e^u}{u^2+v^2}\right)$

$T_u = \left(1, 0, \frac{e^u}{u^2+v^2} - \frac{2ue^u}{(u^2+v^2)^2}\right)$ $T_v = \left(0, 1, \frac{-2ve^u}{(u^2+v^2)^2}\right)$. At $(u,v) = (0,1)$

$T_u = (1, 0, 1)$ $T_v = (0, 1, -2)$, $T_u \times T_v = (-1, 2, 1)$

The tangent plane is $-1(x-0) + 2(y-1) + (z-1) = 0$, $-x + 2y + z - 3 = 0$

2. (35 points) Consider the function $f(x, y) = (2 + \sin(x\,y))\, e^{2y^2}$. Answer the following two questions.

(a) (20 points) Find all the critical points of f and classify them.

We solve $\frac{\partial f}{\partial x} = y\cos(xy)\, e^{2y^2} = 0$ and $\frac{\partial f}{\partial y} = 8y\, e^{2y^2} + x\cos(xy)\, e^{2y^2} = 0$

Eqn 1 says $y = 0$ or $\cos(xy) = 0$, $xy = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Eqn 2 says $8y + x\cos(xy) = 0$. The possible solutions $xy = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ to Eqn 1 do not solve Eqn 2. Thus $y = 0$, in which case $x = 0$ from Eqn 2. $(x, y) = (0, 0)$ is the only critical point.

$\frac{\partial^2 f}{\partial x^2} = -y^2 \sin(xy)\, e^{2y^2} = 0$ at $(0,0)$. $\frac{\partial^2 f}{\partial y\,\partial x} = \cos(xy)e^{2y^2} - xy\sin(xy)e^{2y^2} + 4y^2\cos(xy)\, e^{2y^2} = 1$ at $(0,0)$

$$\frac{\partial^2 f}{\partial y^2} = 8e^{2y^2} + 32y^2e^{2y^2} - x^2\sin(xy)e^{2y^2} + 4yx\cos(xy)e^{2y^2}$$
$= 8$ at $(0,0)$

$H = \begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix}$. $f_{yy} \neq 0$ and $\det H = -1$, so $(0,0)$ is a saddle point.

(b) (15 points) Find the second-order Taylor approximation to f at the point $(0,0)$ and use it to approximate $f(0.1,-0.01)$.

Solution $f(0,0) = 2$, $f_x(0,0) = f_y(0,0) = 0$ $H = \begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix}$ so the second order Taylor approximation is

$$g(x,y) = 2 + xy + \frac{8}{2} y^2$$

$$g(0.1,-0.01) = 2 - 0.001 + 4 \cdot 0.0001 = 1.9994$$

3. (35 points) A particle moves with constant velocity, starting at the point $(1, 1, 1)$ in outward normal direction of the surface $x^2 + 2y^2 + 2z^2 = 5$ at a speed of 3 units per second. At what time does it cross the sphere $x^2 + y^2 + z^2 = 19$?

The normal direction is $\nabla(x^2+2y^2+2z^2) = (2x, 4y, 4z)$ and at $(1,1,1)$ this is $(2,4,4)$. The unit normal is $\frac{1}{\sqrt{4+16+16}}(2,4,4)$ $= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$. Thus the position of the particle at time t is

$$c(t) = (1,1,1) + 3t\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = (1+t, 1+2t, 1+2t)$$

We solve $(1+t)^2 + (1+2t)^2 + (1+2t)^2 = 19$

$$9t^2 + 10t - 16 = 0, \qquad t = \frac{-10 \pm \sqrt{100 + 576}}{18} = \frac{-10 \pm 26}{18}$$

The positive root is $\frac{16}{18} = \frac{8}{9}$, so $t = \frac{8}{9}$ seconds.

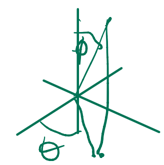
4. (30 points) Let S be the closed surface enclosing the portion of the ball $x^2 + y^2 + z^2 \leq 1$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and oriented with outward unit normal. Calculate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}, \qquad \text{where} \quad \mathbf{F}(x,y,z) = (-xyz,\, y^2z + x,\, e^x).$$

Use Gauss's theorem: $\nabla \cdot F = -yz + 2yz = yz$. Thus

$\iint_S F \cdot dS = \iiint_V yz\, dV$ where V is the region enclosed by S, noting that S is correctly oriented.

Spherical coordinates give $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \cos\phi \sin\phi \sin\theta \; \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$



$z = \rho\cos\phi$

$y = \rho \sin\phi \sin\theta$

$= \left[\dfrac{\rho^5}{5}\right]_0^1 \left[\dfrac{\sin^3\phi}{3}\right]_0^{\frac{\pi}{2}} \left[-\cos\theta\right]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{5} \cdot \dfrac{1}{3} \cdot 1 = \dfrac{1}{15}$

5. (35 points) Let $\mathbf{F}(x, y, z) = (e^x + yz,\, xz,\, xy + 3z^2)$. Calculate

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{c}(t) = (\sin^3 t,\, \cos^5 t,\, \cos^7 t)$ with $0 \le t \le \pi$.

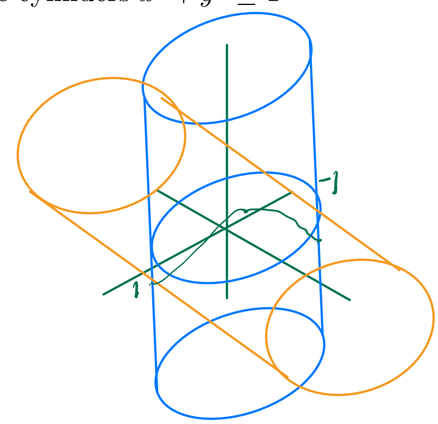
Solution. We quickly check $\nabla \times F = 0$ and find f so that

$\frac{\partial f}{\partial x} = e^x + yz$, $\frac{\partial f}{\partial y} = xz$, $\frac{\partial f}{\partial z} = xy + 3z^2$. $f(x,y,z) = e^x + z^3 + xyz$

Now $c(0) = (0,1,1)$ $c(\pi) = (0,-1,-1)$ so

$\int_c F \cdot ds = f(0,-1,-1) - f(0,1,1) = (1-1+0) - (1+1+0) = -2$.

6. (30 points) Find the volume of the region given by the intersection of the cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$.

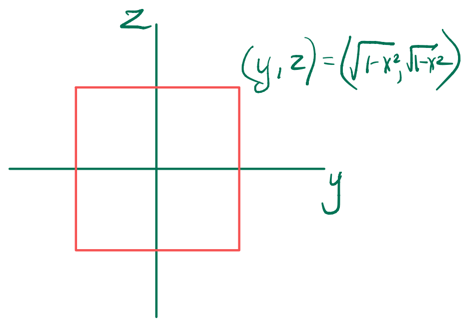


The cross-sections perpendicular to the x-axis are squares with vertices at $\left(\pm\sqrt{1-x^2}, \pm\sqrt{1-x^2}\right)$, giving a square of area $\left(2\sqrt{1-x^2}\right)^2 = 4(1-x^2)$.

The volume is thus $\displaystyle\int_{-1}^{1} 4(1-x^2)\,dx$

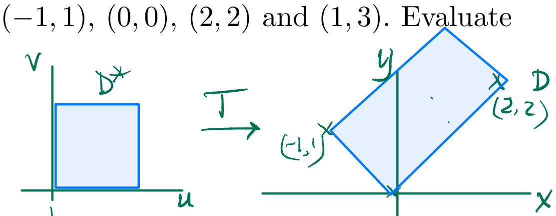
$$= \left[4x - \frac{4x^3}{3}\right]_{-1}^{1} = 4 - \frac{4}{3} - \left(-4 + \frac{4}{3}\right)$$

$$= 8 - \frac{8}{3} = \frac{16}{3} = 5\frac{1}{3}$$



7. (30 points) Let D be the parallelogram with vertices $(-1,1)$, $(0,0)$, $(2,2)$ and $(1,3)$. Evaluate the double integral

$$\iint_D x\,dA.$$



The mapping $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by the matrix $\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$ sends the unit square $[0,1]\times[0,1]$ to D

Thus $T\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u - v \\ 2u + v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and if $f(x,y) = x$ then $f\left(T\begin{bmatrix} u \\ v \end{bmatrix}\right) = 2u - v$.

Also $\dfrac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\right| = 4$

Thus $\displaystyle\iint_D x\,dx\,dy = \iint_{D^*} 2u - v \,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \int_0^1\!\!\int_0^1 8u - 4v\,du\,dv$

$\displaystyle = \int_0^1 \left[4u^2 - 4uv\right]_0^1 dv = \int_0^1 (4 - 4v)\,dv = \left[4v - 2v^2\right]_0^1 = 2$

8. (25 points) Let $g(x, y, z) = x^3 + 5yz + z^2$ and let $h(u)$ be a function of one variable such that $h'(1) = 1/2$. Let $f = h \circ g$. Starting at $(1, 0, 0)$, in what directions is f changing at 50% of its maximum rate?

Solution: The chain rule says $(Df)(1,0,0) = Dh\,(f(1,0,0)) \circ Dg(1,0,0)$

$= Dh(1)\ Dg(1,0,0) = \frac{1}{2} Dg(1,0,0)$.

$Dg = [3x^2, 5z, 5y+2z]$ so $Dg(1,0,0) = [3, 0, 0]$ and $Df(1,0,0) = [\frac{3}{2}, 0, 0]$

f is changing most rapidly in the direction of its gradient, which is in direction $(1,0,0)$, and its rate of change is $\frac{3}{2}$ in that direction.

We find unit vectors (u, v, w) so that $(3/2, 0, 0) \cdot (u, v, w) = \frac{3}{4}$, i.e.

$\frac{3}{2} u = \frac{3}{4}$, $u = \frac{1}{2}$. $u^2 + v^2 + w^2 = 1$ means $v^3 + w^2 = \frac{3}{4}$, so (u, v, w) lie on a circle center $(\frac{1}{2}, 0, 0)$ radius $\frac{\sqrt{3}}{2}$.

9. (30 points) Calculate

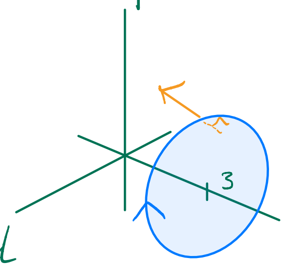
$$\int_{\mathbf{c}} (x^3y + e^z)\,dx + (y^3z + e^x)\,dy + (xe^z + xy)\,dz,$$

where $\mathbf{c}(t) = (2\cos t, 3, 2\sin t)$ with $0 \le t \le 2\pi$.

C is a closed curve. Let S be the disk that has boundary C, oriented as shown by the unit normal $(0,-1,0)$. S is a circle of radius 2

By Stokes' theorem the integral is

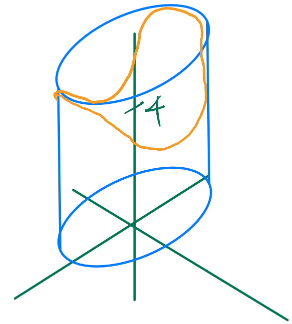
$$\iint_S \left(x - y^3,\ e^z - e^z - y,\ e^x - x^3\right) \cdot (0,-1,0)\, dS$$



$$= \iint_S y\, dS$$

. Now $y = 3$ on S, so the integral is $3 \cdot$ Area of S $= 3\pi 2^2 = 12\pi$.

10. (25 points) Compute the area of the portion of the cylinder $x^2 + y^2 = 1$ that lies between $z = 0$ and $z = 4 + x^2 - y^2$.



Cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$

so $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$

Here $r = 1$, and we can parametrize the surface by

$\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$ $\quad 0 \le \theta \le 2\pi$, $0 \le z \le 4 + \cos 2\theta$

$T_\theta = (-\sin\theta, \cos\theta, 0)$, $T_z = (0, 0, 1)$, $T_\theta \times T_z = (\cos\theta, \sin\theta, 0)$

$\|T_\theta \times T_z\| = 1$.

The area is $\displaystyle\int_0^{2\pi}\int_0^{4+\cos 2\theta} 1 \, dz\, d\theta = \int_0^{2\pi} (4 + \cos 2\theta)\, d\theta = \left[4\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}$

$= 8\pi$.

Or :